# RATIO LIMIT THEOREMS FOR MARKOV PROCESSES

BY

S. R. FOGUEL

ABSTRACT

Convergence of $\sum_{n=0}^{N}\mu P^n(B)/\sum_{n=0}^{N}\mu P^n(A)$ and $\mu P^n(B)/\mu P^n(A)$ is established for a certain class of Markov operators, $P$, where $\mu$ is a measure and $B$ is a subset of $A$. The results are proved under certain conditions on $P$ and the set $A$.

**Definitions and notations.** Let $(X,\Sigma,m)$ be a measure space, $m(X)=1$. Let $P$ be an operator on $L_1(X,\Sigma,m)$ that satisfies:

(1) If $u \geqq 0$ a.e. then $uP \geqq 0$ a.e.

(2) $\int |uP|\,dm \leqq \int |u|\,dm$.

The operator $P$ is defined on $L_\infty(X,\Sigma,m)$ by $\langle u, Pf\rangle = \langle uP, f\rangle$.

By a charge $\tau$ we mean a non-negative, *finitely* additive finite measure that is weaker than $m$ i.e. if $m(A)=0$ then $\tau(A)=0$. The operator $P$ is defined on charges by $\tau P(A)=\int P1_A d\tau$. This same equation is used to define $P$ on $\sigma$ finite measure, weaker than $m$. Note that if $\lambda$ is a $\sigma$ finite measure then $\lambda P$ need not be $\sigma$ finite. If the charge $\tau$ is a measure (is countably additive) and $u = d\tau/dm$ then $uP = d(\tau P)/dm$ and in particular $\tau P$ is a measure again.

A charge $\tau$ is called a *pure charge* provided:

*If $\mu$ is a measure and $\mu \leqq \tau$ then $\mu = 0$.*

Note that we use only non-negative charges. A bounded finitely additive measure weaker than $m$, which is not necessarily non-negative, will be called a functional on $L_\infty$.

The following theorem of Yosida and Hewitt (see [8], theorem 1.22 and [1], chap. IV, lemma A) will be used often:

THEOREM *Every charge $\tau$ can be decomposed uniquely into the sum $\tau_1+\tau_2$*

Received August 3, 1969.

*where $\tau_1$ is a measure and $\tau_2$ a pure charge. There exists a sequence of sets, $X_n$, such that $m(X_n) \to 1$ and $\tau_2(X_n) = 0$.*

Throughout this paper we shall assume that $P$ is ergodic and conservative, i.e.:

$$\textit{If } m(A) > 0 \textit{ then } \sum_{n=0}^{\infty} P^n 1_A \equiv \infty.$$

Details of the various definitions given here can be found in [1].

1. **The ratio limit theorem.** Throughout this paper we shall assume:

CONDITION I. *There exists a set $A$, with $m(A) > 0$, such that if $m(B) > 0$ then $\sum_{n=0}^{N} P^n 1_B \geqq \eta 1_A$ where $N = N(B)$ and $0 < \eta = \eta(B)$.*

Note that by $1_E$ we denote the characteristic function of $E$. Also every set used is a measurable set.

REMARKS ON CONDITION I. (1) If Condition I holds for a set $A$ then it holds for any subset of A.

(2) Let $\varepsilon > 0$ be given and assume Condition I for "big sets" only, namely for sets $B$ such that $m(B) > 1 - \varepsilon$. Then Condition I holds: Let $E$ be a set with $m(E) > 0$. Put $B = \{x\colon \sum_{k=0}^{K} P^k 1_E \geqq 1\}$. If $K$ is large enough, then $m(B) > 1 - \varepsilon$ since $P$ is ergodic and conservative. Thus

$$\eta 1_A \leqq \sum_{n=0}^{N} P^n 1_B \leqq \sum_{n=0}^{N} P^n \sum_{k=0}^{K} P^k 1_E \leqq K \sum_{j=0}^{N+K} P^j 1_E$$

or

$$N(E) = N(B) + K \text{ and } \eta(E) = \frac{\eta(B)}{K}.$$

(3) Put $A_1 = \{x\colon \sum_{k=0}^{K} P^k 1_A \geqq 1\}$. As $K \to \infty$ $m(A_1) \to 1$. Now

$$\eta 1_{A_1} \leqq \eta \sum_{k=0}^{K} P^k 1_A \leqq \sum_{k=0}^{K} P^k \sum_{n=0}^{N} P^n 1_B \leqq K \sum_{j=0}^{N+K} P^j 1_B.$$

This Condition I holds for the set $A_1$ as well.

LEMMA 1. *If Condition I holds then every invariant pure charge, $\tau$, vanishes on $A$.*

**Proof.** By the Yosida-Hewitt Theorem there exists a set $B$ with $m(B) > 0$ and $\tau(B) = 0$. Now, since $\tau$ is invariant,

$$0 = \left\langle \tau, \sum_{n=0}^{N} P^n 1_B \right\rangle \geqq \eta \tau(A).$$

If $A = X$ in Condition I, one can conclude:

LEMMA 2. *Assume that the only invariant pure charge is zero. There exists a unique invariant charge, $\lambda$, with $\lambda(X) = 1$. The invariant charge $\lambda$ is a measure.*

**Proof.** Let $\tau$ be an invariant charge and $\tau = \tau_1 + \tau_2$ its Yosida-Hewitt decomposition. Now $\tau = \tau P = \tau_1 P + \tau_2 P$ and $\tau_1 P$ is a measure while $\tau_2 P$ can be decomposed again. Thus $\tau_1 P \leqq \tau_1$ and by [1], (2.10) equality holds, therefore $\tau_2 P = \tau_2$ and by assumption $\tau_2 = 0$. Hence every invariant charge is a measure. An invariant charge exists since the set of charges $\mu$ with $\mu(X) = 1$ is a weak * com. pact and convex set invariant under $P$. Uniqueness of the invariant measure follow from [1] chap. VI, theorem A, since $P$ is ergodic and conservative.

Under the condition of the Lemma, one can show that every invariant functional is a multiple of $\lambda$. This involves showing that the positive and negative parts of an invariant functional are invariant charges. Hence, by the Hahn Banach Theorem, the range of $I - P$ is dense in the subspace of $L_\infty$: $\{f : \langle \lambda, f \rangle = 0\}$. Thus for every $f \in L_\infty$

$$\text{ess. sup} \left| \frac{1}{N+1} \sum_{n=0}^{N} P^n f - \langle \lambda, f \rangle \right| \underset{N \to \infty}{\to} 0$$

which implies Condition I with $A = X$.

If $A$ is not equal to $X$ and there exists an invariant measure, $\lambda$, for $P$ (which is necessarily unique) then if $f$ is supported on $A$ and $\langle \lambda, f \rangle = 0$ then every invariant charge vanishes on $f$ and again ess. sup$| 1(/1 + 1) \Sigma_{n=0}^{N} p^n f|_{N \to \infty} \to 0$. This generalizes Theorem 2 of S. Horowitz *$L_\infty$-Limit theorems for Markov processes*, Israel J. of Math. 7 (1969), 60–62, since by Section III if $P$ is a Harris process Condition I is satisfied for some set $A$.

If $A$ is not equal to $X$ let us follow Harris [2] to localize the process to $A$. Define the operator $T_E$ on $L_1(X, \Sigma, m)$ by:

$$uT_E(x) = 1_E(x) \cdot u(x).$$

Also define the operator $P_A = \Sigma_{n=0}^{\infty} (PT_{A'})^n PT_A$ (convergence in the strong sense). Then $(A, \Sigma, m, P_A)$ is a Markov process and $P_A 1 = P_A 1_A = 1$. See [1], chap. VI, lemma B. The following Lemma is again due to Harris but we will prove it for completeness sake.

LEMMA 3. *If $f \geqq 0$ is supported on $A$ then*

$$\sum_{n=0}^{N} P^n f \leqq \sum_{n=0}^{N} P_A^n f.$$

**Proof.** For every $0 \leqq n \leqq N$ $P^n f = (PT_A + PT_{A'})^n f$ is the sum of expressions of the type $(PT_A)^{i_0}(PT_{A'})^{i_1} \cdots (PT_A)^{i_r} f$ where $i_k$ are non-negative integers and $i_0 + i_1 + \cdots + i_r = n$. Also $i_r > 0$ since $T_{A'} f = 0$. Now for every $0 \leqq k \leqq N$

$$P_A^k f \geqq [PT_A + \cdots + (PT_{A'})^N PT_A]^k f.$$

Take in this product the first term $i_0$ times and multiply it, on the right, by $(PT_{A'})^{i_1} PT_A$ and then again by $PT_A$ $i_2 - 1$ times and so on. This is possible if

$$k = i_0 + 1 + i_2 - 1 + \cdots + 1 + i_r - 1 = i_0 + i_2 + \cdots + i_r \leqq n \leqq N.$$

Thus every term on the left hand side of the inequality is dominated by an appropriate term on the right hand side.

From Lemma 3 follows that $P_A$ is again conservative and ergodic and satisfies Condition I. Now $P_A$ acts on $(A, \Sigma, m)$ and thus Lemma 2 applies.

Definition. *Let $\tilde{\lambda}$ be the unique invariant charge of $P_A$ such that $\tilde{\lambda}(A) = 1$. Put $\lambda = \sum_{n=0}^{\infty} \tilde{\lambda}(PT_{A'})^n$.*

The set function $\lambda$ is a $\sigma$ finite measure invariant for $P$ (see [1], chap. VI, theorem C). Also if $f$ is supported on $A$ then

$$\langle \lambda, f \rangle = \tilde{\lambda}, \left\langle \sum_{n=0}^{\infty} (PT_{A'})^n f \right\rangle = \langle \tilde{\lambda}, f \rangle.$$

The fact that Condition I is a sufficient condition for the existence of a $\sigma$ finite invariant measure is a corollary of a result of Horowitz see [3], theorem 1.

Theorem 1. *Let $P$ satisfy Condition I. If $0 \leqq f \in L_\infty$ is supported on $A$ and $\mu$ is a measure then*

$$\lim \frac{\sum_{n=1}^{N} \langle \mu P^n, f \rangle}{\sum_{n=1}^{N} \mu P^n(A)} = \frac{\langle \lambda, f \rangle}{\lambda(A)}.$$

**Proof.** It is enough to prove that if $N_j$ is a subsequence of the integers and

$$\frac{\sum_{n=1}^{N_j} \langle \mu P^n, f \rangle}{\sum_{n=1}^{N_j} \mu P^n(A)}$$

converges then the limit is equal to $\langle\lambda,f\rangle/\lambda(A)$. Denote a Banach limit by LIM and define the functional $\tau$ over $L_\infty$ by

$$\langle\tau,g\rangle = \mathrm{LIM}\ \frac{\sum_{n=1}^{N_j}\langle\mu P^n, T_A g\rangle}{\sum_{n=1}^{N_j}\mu P^n(A)}.$$

Clearly $\tau$ is a charge and from [3], lemma 2 it follows that $\langle\tau,P_A g\rangle \leqq \tau,g\rangle$ for every $0 \leqq g \in L_\infty$. Now since $P_A 1 = 1$ the charge $\tau$ is invariant for $P_A$ and by Lemma 3 $\tau = \tilde{\lambda}$. Finally, our results follows from $T_A f = f$ and $\langle\tilde{\lambda},f\rangle = \langle\lambda,f\rangle$.

REMARK. In [4] Horowitz proved a similar result. There, Condition I is not assumed but $X$ is a topological space, $A$ a compact set and $Pf$ is continuous whenever $f$ is. The proof presented here is an adaptation of Horowitz's proof.

2. **The strong ratio limit theorem.** Let us assume in this section

CONDITION II. *There exists a measure $\mu$ such that*

$$\lim\frac{\mu P^n(B) - \mu P^{n-1}(B)}{\mu P^n(A)} = 0$$

*for every $B \subset A$.*

Eventually we wish to prove that $\lim(\langle\mu P^n,f\rangle/\langle\mu P^n,g\rangle) = (\langle\lambda,f\rangle/\langle\lambda,g\rangle)$ and since $\lambda P = \lambda$ Condition II is clearly necessary. Note that $\lim(\mu P^{n-1}(A)/\mu P^n(A)) = 1$. Define

$$M = \{f : f \in L_\infty \textit{ and } \frac{\langle\mu,P^n f\rangle - \langle\mu P^{n-1},f\rangle}{\mu P^n(A)} \to 0\}.$$

LEMMA 4. *The set $M$ is linear and $PM \subset M$. Every $f \in L_\infty$ which is supported on $A$ belongs to $M$. For every integer $k$ and every $f \in L_\infty (PT_{A'})^k PT_A f \in M$.*

**Proof.** Linearity of $M$ is obvious. Let $f \in M$, then

$$\lim\frac{\langle\mu P^n,Pf\rangle - \langle\mu P^{n-1},Pf\rangle}{\mu P^n(A)}$$

$$= \lim\frac{\langle\mu P^{n+1},f\rangle - \langle\mu P^n,f\rangle}{\mu P^{n+1}(A)}\lim\frac{\mu P^{n+1}(A)}{\mu P^n(A)} = 0$$

by Condition II. Now if $f$ is supported on $A$ then for every $\varepsilon > 0$ one can find a step function $\Sigma b_i 1_{B_i}$ where $B_i \subset A$ and $\|f - \Sigma_i b_i 1_{B_i}\|_\infty \leqq \varepsilon 1_A$. Thus, by Condition II,

$$\limsup\left|\frac{\langle\mu P^n,f\rangle - \langle\mu P^{n-1},f\rangle}{\mu P^n(A)}\right| \leqq \varepsilon.$$

Let us show that $(PT_{A'})^k PT_A f \in M$ by induction on $k$. If $k = 0$ then $T_A f \in M$ hence $PT_A f \in M$ too. Now

$$(PT_{A'})^{k+1} PT_A f = P[(PT_{A'})^k PT_A f - T_A(PT_{A'})^k PT_A f]$$

the first term, in the brackets, belongs to $M$ by the induction hypothesis and the second term is supported on $A$ and thus belongs to $M$.

Let us introduce now another condition:

CONDITION III. $\| T_A(PT_{A'})^N 1 \|_{N\to\infty} \to 0$.

Now for every $f \in L_\infty$

$$\left| P_A f - \sum_{n=0}^{N} (PT_{A'})^n PT_A f \right| = \sum_{n=N+1}^{\infty} (PT_{A'})^n PT_A f \leqq \|f\|_\infty (PT_{A'})^{N+1} 1.$$

Thus our condition is equivalent to the operator norm convergence of $T_A \sum_{n=0}^{N} (PT_{A'})^n PT_A$ to $T_A P_A$.

THEOREM 2. *Assume Conditions I, II and III. If* $0 \leqq f \in L_\infty$ *is supported on A then*

$$\lim \frac{\langle \mu P^n, f\rangle}{\mu P^n(A)} = \frac{\langle \lambda, f\rangle}{\lambda(A)}.$$

**Proof.** It is enough to show that if the subsequence $\langle \mu P^{n_i}, f\rangle / \mu P^{n_i}(A)$ converges then the limit is $\langle \lambda, f\rangle / \lambda(A)$. Define the functional

$$\langle \tau, g\rangle = \mathrm{LIM} \frac{\langle \mu P^{n_i}, T_A g\rangle}{\mu P^{n_i}(A)}$$

and as in Theorem 1, it is enough to show that $\tau P_A = \tau$. Let $g \geqq 0$ then

$$P^n T_A \sum_{k=0}^{K} (PT_{A'})^k PT_A g = P^n(I - T_{A'}) \sum_{k=0}^{K} (PT_{A'})^k PT_A g$$

$$= P^n \sum_{k=0}^{K} (PT_{A'})^k PT_A g - P^{n-1} \sum_{k=1}^{K+1} (PT_{A'})^k PT_A g$$

$$= P^n \sum_{k=0}^{K} (PT_{A'})^k PT_A g - P^n \sum_{k=1}^{K+1} (PT_{A'})^k PT_A g + (P^n - P^{n-1}) \sum_{k=1}^{K+1} (PT_{A'})^k PT_A g$$

$$= P^{n+1} T_A g - P^n (PT_{A'})^{K+1} PT_A g + (P^n - P^{n-1}) \sum_{k=1}^{K+1} (PT_{A'})^k PT_A g$$

$$\leqq P^n T_A g + (P^n - P^{n-1}) \sum_{k=0}^{K+1} (PT_{A'})^k PT_A g.$$

Now since $\sum_{k=0}^{K+1}(PT_{A'})^kPT_Ag \in M$ by Lemma 4, $\langle \tau, \sum_{k=0}^{K}(PT_{A'})^kPT_Ag\rangle \leqq \langle \tau, g\rangle$. Finally, by Condition III, the sum converges uniformly to $P_Ag$ hence $\langle \tau, P_Ag\rangle \leqq \langle \tau, g\rangle$. Since $\langle \tau, P_A1\rangle = \langle \tau, 1\rangle$ equality holds for every $g \geqq 0$ hence $\tau P_A = \tau$.

3. **Verification of Conditions I and III.** If the set $A$ is an atom, then Conditions I and III are always fulfilled. Let us prove that if $P$ is a Harris process then one can choose a set $A$ so that Conditions I and III hold. For the definition of a Harris process, see [1], chap. V. Now $P^n = Q_n + R_n$ where $Q_nf(x) = \int q_n(x,y)f(y)m(dy)$ and $q_n(x,y)$ is measurable in both variables, and for some $k$ $q_k(x,y)$ is not equal to zero a.e. In order to establish Condition I, let us follow [3]:

For some $\delta > 0$

$$0 < m^2\{(x,y) : q_k(x,y) \geqq \delta\} = \int m(E_x)m(dx)$$

where

$$E_x = \{y : q_k(x,y) \geqq \delta\}.$$

Thus for sore $\varepsilon > 0$ $m(A) > 0$ where $A = \{x : m(E_x) \geqq \varepsilon\}$. Now if $m(B) \geqq 1 - \varepsilon/2$ then, for every $x \in A$, $m(B \cap E_x) \geqq \varepsilon/2$ hence

$$P^k 1_B(x) \geqq \int_B q_k(x,y)m(dy) \geqq \int_{B\cap E_x} q_k(x,y)m(dx) \geqq \delta\varepsilon/2\, 1_A(x)$$

hence Remark (2) after Condition I applies.

It is more difficult to establish Condition III. Let $k$ be chosen as above and put $\tilde{q}(x,y) = \min[q_k(x,y), 1]$ again $\tilde{q}$ is not zero z.e. Put

$$\tilde{Q}f(x) = \int \tilde{q}(x,y)f(y)m(dy)$$

and $\tilde{R} = P^k - \tilde{Q}$. Now $T_A(PT_{A'})^n1$ is monotonically decreasing so it is enough to how that $\| T_A(PT_{A'})^{nk}1 \| \to 0$. To simplify notation we shall assume now that $k = 1$. The main property of $\tilde{Q}$ is (an observation due to Horowitz): *If $\tau$ is a charge then $\tau\tilde{Q}$ is a measure: If $E_n \downarrow 0$ then*

$$\tau\tilde{Q}(E_n) \leqq \sup_x \int_{E_n} \tilde{q}(x,y)m(dy) \leqq m(E_n) \to 0.$$

Now put $P^n = \tilde{Q}_n + \tilde{R}^n$ and note that $\tilde{Q}_n$ is a product of terms that at least one of them is $\tilde{Q}$. Thus $\tilde{Q}_n$, again, maps a charge into a measure.

LEMMA 5. *Let $E_n$ be a sequence of sets decreasing to the null set. For every $k$ $\| T_A\tilde{Q}_k1_{E_n}\|_\infty \to_{n\to\infty} 0$.*

**Proof.** Assume, to the contrary, that for some $\delta > 0$ the sets

$$F_n = \{x: T_A\tilde{Q}_k 1_{E_n}(x) \geqq \delta\}$$

are not empty. Let $v_n$ be measures on $F_n$ $v_n(F_n) = 1$. Now if $n > m$ $v_n\tilde{Q}_k(E_m) \geqq v_n\tilde{Q}_k(E_n) \geqq \delta$ thus if $v$ is a weak * limit of the sequence $v_n$ then $v\tilde{Q}_k(E_m) \geqq \delta$ for every $m$ hence $v\tilde{Q}_k$ is not a measure.

Another useful property of the decomposition of $P^n$ defined above is:

LEMMA 6. *The sequence $\tilde{R}^n 1$ converges monotonically to zero.*

**Proof.** If $\lim \tilde{R}^n 1 = g$ then $g = \tilde{R}^k g \leqq P^k g$ for every $k$ thus equality must hold by [1], chapter II, theorem B, and $\tilde{Q}_k g = 0$ but $\tilde{Q}_{n+k} \geqq \tilde{Q}_k P^n$ hence $0 = \Sigma_{n=0}^{\infty} \tilde{Q}_k P^n g = \tilde{Q}_k \Sigma_{n=0}^{\infty} P^n g$ but $\Sigma P^n g = \infty$ unless $g = 0$ and $\tilde{Q}_k$ is not the zero operator.

Let us now choose $A$ so that $\| T_A \tilde{R}^n 1 \|_\infty \to 0$. Since Condition I is valid whenever one reduces the set $A$, we do not affect the validity of Condition I by this additional hypothesis.

Now

$$T_A(PT_{A'})^j(PT_{A'})^n 1 = T_A(PT_{A'})^j[T_{B_n}(PT_{A'})^n 1 + T_B'(PT_{A'})^n 1]$$

where $B_n$ will be chosen later. Thus

$$T_A(PT_{A'})^{j+n} 1 \leqq \sup T_{B_n}(PT_{A'})^n 1(x) + T_A\tilde{R}^j 1(x) + T_A\tilde{Q}_j 1_{B'_n}.$$

Choose $j$ so large that the middle term will be smaller than $\varepsilon$. Choose $B_n = \{x: (PT_{A'})^n 1(x) < \varepsilon\}$ then the first term is smaller than $\varepsilon$ and the last term tends to zero as $n \to \infty$ by Lemma 5.

Let us conclude with some references.

For Harris' processes a stronger result than Theorem 1 was proved in [6].

The strong ration limit theorem (Theorem 2) for Harris' processes was proved in [5] under different assumptions.

For matrices Theorem 2 irplies (by taking $A = \{j, k\}$ and $\mu$ a unit measure at $\{i\}$ that:

$$\text{if } \frac{p_{i,j}^{(n)} - p_{i,j}^{(n-1)}}{p_{i,j}^{(n)} + p_{i,k}^{(n)}} \to 0 \text{ and } \frac{p_{i,k}^{(n)} - p_{i,k}^{(n-1)}}{p_{i,j}^{(n)} + p_{i,k}^{(n)}} \to 0$$

then $$\frac{p_{ij}^{(n)}}{p_{i.k}^{(n)}} \text{ converges.}$$

In (7) Orey proved that $\dfrac{p_{ij}^{(n)}}{p_{ii}^{(n)}}$ converges provided $\dfrac{p_{ii}^{(n)} - p_{ii}^{(n-1)}}{p_{ii}^{(n)}} \to 0.$

## References


1. S. R. Foguel, *The ergodic theory of Markov processes*, Van Nostrand, 1969.

2. T. E. Harris, *The existence of stationary measures for certain Markov processes*, Third Berkeley Symp. Math. Stat. Prob. **2** (1956), 113–124.

3. S. Horowitz, *On $\tau$ finite invariant measures for Markov processes*, Israel J. Math. **6** (1968), 338-345.

4. S. Horowitz, *Markov processes on a locally compact space*, to be published.

5. N. C. Jain, *The strong ratio limit property for some general Markov processes*, Ann. Math. Statist. **40** (1969), 986–992.

6. N. C. Jain, *Some limit theorems for a general Markov process*, Z. Wahrscheinlishkeits-theorie und Verlw. Gebiete **6** (1966), 206–223.

7. S. Orey, *Strong ratio limit property*, Bull. Amer. Math. Soc. **67** (1961), 571–574.

8. K. Yosida and E. Hewitt, *Finitely additive measures*, Trans. Amer. Math. Soc. **72** (1952), 46–66.


Hebrew University of Jerusalem